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Some curious Dirichlet series

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Problems on analytic continuation of "number theoretical" Dirichlet series constitute an important part of analytic number theory. There are some well known methods such as Poisson summmation formula, Euler-Maclaurin formula, contour integration, etc.. In this paper we discuss some marginal Dirichlet series for which the above methods seems to be useless.

1 Fibonacci zeta

Let $\{F_n\}_{n\in N}$ be the Fibonacci sequence defined by $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n, (n \geq 3)$. Define

$$Fib(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}. \tag{1}$$

It is easily seen that the above series converges absolutely and uniformly in the wide sence in the region $\Re s > 0$. Recently the problem of transcendency of $Fib(s)$ at positive integers became reachable by the advances of the theory of Mahler functions.

Proposition 1 (D. Deverney, Ke. Nishioka, Ku. Nishioka, I. Shiokawa) *For even positive integers s, $Fib(s)$ are transcendental.*

The corresponding results for odd integers(containing 1) are conjectured but unsolved([1]). The results have a curious similarity to the corresponding ones for the Riemann zeta functions. So, it is of some interest to discuss the analytic properties of $Fib(s)$ as function of variable s. We have

Theorem 1 *$Fib(s)$ can be continued to a meromorphic function holomorphic exept the simple poles at $s = -2k + \frac{\pi i n}{\log \alpha}$, where k is a non-negative integer, n in an integer, and $\alpha = \frac{1+\sqrt{5}}{2}$.*

The proof is quite easy. From the well known expression of Fibonacci numbers:

$$F_n = \frac{\alpha^n - (-\alpha)^{-n}}{\sqrt{5}} \tag{2}$$

we have

$$\begin{aligned} Fib(s) &= \sqrt{5}^s \sum_{n=0}^{\infty} \frac{1}{(\alpha^n - (-\alpha)^{-n})^s} \\ &= \sqrt{5}^s \left(\sum_{m=1}^{\infty} \frac{1}{(\alpha^{2m} + \alpha^{-2m})^s} + \sum_{m=0}^{\infty} \frac{1}{(\alpha^{2m+1} - \alpha^{-2m-1})^s} \right) \\ &= \sqrt{5}^s \left(\sum_{m=1}^{\infty} \frac{1}{\alpha^{2ms}} (1 + \alpha^{-4m})^s + \sum_{m=0}^{\infty} \frac{1}{\alpha^{(2m+1)s}} (1 - \alpha^{-(4m+2)})^s \right) \\ &= \sqrt{5}^s \, (g(s) + f(s)) \, (set) \end{aligned}$$

From the binary expansion and the sum formula of geometric series we have

$$g(s) = \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{\alpha^{2s+4k} - 1} \tag{3}$$

and

$$f(s) = \sum_{k=0}^{\infty} \binom{-s}{k} \frac{\alpha^{-s-2k}}{1 - \alpha^{-2s-4k}} \tag{4}$$

which give meromorphic continuation.

2 Additive convolution

Let

$$f_1(s) = \sum_{n=1}^{\infty} \frac{a_1(n)}{n^s}, \ldots, f_r(s) = \sum_{n=1}^{\infty} \frac{a_r(n)}{n^s} \tag{5}$$

be Dirichlet series with "nice" properties. What properties are inherited by their "additive convolution" i.e.

$$f_1 \diamond \cdots \diamond f_r(s) = \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{a_1(n_1) \cdots a_r(n_r)}{(n_1 + \cdots + n_r)^s}. \tag{6}$$

Note that, when f_i are the Riemann zeta function the additive convolution become r-ple zeta function of Barnes. This example suggest the analytic continuability of $f_1 \diamond \cdots \diamond f_r(s)$ in general. In fact we can prove

Theorem 2 *If $f_i(s), (i = 1, \ldots, r)$ can be continued to meromorphic functions holomorphic exept finite number of poles and of polynomial growth along the imaginary direction, then so is their additive convolution $f_1 \diamond \cdots \diamond f_r(s)$*

The proof is based on the following integral expression:

Lemma 1

$$\sum_{n_1,\dots,n_r=1}^{\infty} \frac{a_1(n_1)\cdots a_r(n_r)}{(n_1+\cdots+n_r+1)^s} =$$

$$\frac{1}{(2\pi i)^r}\int_{(c_1)}\cdots\int_{(c_r)}\frac{\Gamma(s_1)\cdots\Gamma(s_r)\Gamma(s-s_1-\cdots-s_r)}{\Gamma(s)}f_1(s_1)\cdots f_r(s_r)ds_1\cdots ds_r,$$

where c_i is a real number in the absolute convergent region of $f_i(s)$.

Note that the left hand side is not exactly the additive convolution. But the difference can be easily recovered by binary expansion of the denominator. The analytic continuation is done by shift of the lines of integration. The detail of the proof will appear elsewhere.

The same method can be applied successfully even more general Dirichlet series. We discus

$$G(s) = \sum_{n=1}^{\infty}\frac{g(n)}{n^s}, \tag{7}$$

where $g(n) = \sum_{m+k=n}\Lambda(m)\Lambda(k)$, which is interesting from the viewpoint of additive prime theory(cf. [2]). Though this case does not satisfy the assumptions of **Theorem 2** we have

Theorem 3 *On the assumption of the Riemann hypothesis, G(s) can be continued to a function meromomrphic in the region $\Re s > 1$ and have a expression*

$$G(s) = \frac{2}{(s-2)(s-1)} - \frac{1}{\Gamma(s)}\sum_{\rho}\Gamma(\rho)\Gamma(s-1-\rho) + J(s),$$

where the summation runs over all the non-trivial zeros of the Riemann zeta function, $J(s)$ is a function holomorphic in $\Re s > 1$, and $J(s) = O((\Im s)^{1+\epsilon})$ for any ϵ in this region.

References

[1] D. Duverney, Ke. Nishioka, Ku. Nishioka, I. shiokawa, Transcendence of Roger-Ramanujan continued fraction and reciprocal sums of Fibonacci numbers, RIMS Proceeding, 1060(1998),91-100

[2] A. Fujii, Acta Arith. 1993?